Quantum fields on the Groenewold-Moyal plane: C, P, T and CPT

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ABSTRACT: We show that despite the inherent non-locality of quantum field theories on the Groenewold-Moyal (GM) plane, one can find a class of **C**, **P**, **T** and **CPT** invariant theories. In particular, these are theories without gauge fields or with just gauge fields and no matter fields. We also show that in the presence of gauge fields, one can have a field theory where the Hamiltonian is **C** and **T** invariant while the S-matrix violates **P** and **CPT**.

In non-abelian gauge theories with matter fields such as the electro-weak and QCD sectors of the standard model of particle physics, **C**, **P**, **T** and the product of any pair of them are broken while **CPT** remains intact for the case $\theta^{0i} = 0$. (Here $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$, x^μ: coordinate functions, $\theta^{\mu\nu} = -\theta^{\nu\mu}$ = constant.) When $\theta^{0i} \neq 0$, it contributes to breaking also **P** and **CPT**. It is known that the S-matrix in a non-abelian theory depends on $\theta^{\mu\nu}$ only through θ^{0i}. The S-matrix is frame dependent. It breaks (the identity component of the) Lorentz group. All the noncommutative effects vanish if the scattering takes place in the center-of-mass frame, or any frame where $\theta^{0i} P_i^{\text{in}} = 0$, but not otherwise. **P** and **CPT** are good symmetries of the theory in this special case.

KEYWORDS: Discrete and Finite Symmetries, Space-Time Symmetries, QCD.

Contents

1. Introduction

The Groenewold-Moyal plane or GM plane $\mathcal{A}_\theta(\mathbb{R}^N)$ is the algebra of smooth functions on $\mathbb{R}^N$ with the $\star$-product

$$f \star g = f e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g, \tag{1.1}$$

$$\theta^{\mu\nu} = -\theta^{\nu\mu} = \text{constant}.$$

If $x = (x^0, x^1, \ldots, x^{N-1})$ labels a point on $\mathbb{R}^N$ and $\hat{x}^\mu$ are coordinate functions,

$$\hat{x}^\mu(x) = x^\mu, \tag{1.2}$$

eq. (1.1) implies the commutation relation

$$(\hat{x}^\mu \star \hat{x}^\nu - \hat{x}^\nu \star \hat{x}^\mu) = [\hat{x}^\mu, \hat{x}^\nu]_\star = i\theta^{\mu\nu}. \tag{1.3}$$

Following Drinfel'd's original work [1], Chaichian et al. [3] and Aschieri et al. [4], have shown that the diffeomorphism group $\mathcal{D}(\mathbb{R}^N)$ of $\mathbb{R}^N$ acts on $\mathcal{A}_\theta(\mathbb{R}^N)$ provided its coproduct Δ_θ is the following twisted one:

$$\Delta_\theta(g) = \mathcal{F}_\theta^{-1}(g \otimes g)\mathcal{F}_\theta, \tag{1.4}$$

$$\mathcal{F}_\theta = e^{\frac{i}{2}\partial_\mu \otimes \theta^{\mu\nu}\partial_\nu}. \tag{1.5}$$

where g is the group element and $\mathcal{F}_\theta$ is called the Drienfel'd twist [1].

The Poincaré group $\mathcal{P}$ is a subgroup of $\mathcal{D}(\mathbb{R}^N)$. Previous papers [5, 6, 2, 7, 8], examined quantum field theories (qft's) on the GM plane $\mathcal{A}_\theta(\mathbb{R}^N)$ which are deformations of the Poincaré invariant qft's for $\theta^{\mu\nu} = 0$. It focused on the identity component $\mathcal{P}_+^\uparrow$ of $\mathcal{P}$. Using the twisted coproduct for $\mathcal{P}_+^\uparrow$, the following was proved [3, 4, 9, 17] in a particular approach to gauge theories: i.) in the absence of gauge fields, these theories are Poincaré invariant. ii.) Poincaré invariance is maintained also by abelian gauge theories (with or without matter) and non-abelian gauge theories without matter. iii.) Poincaré invariance is lost in non-abelian gauge theories with matter if $\theta^{0i} \neq 0$ while it is maintained if $\theta^{0i} = 0$.

Parity **P**, time reversal **T** and **PT** are not elements of $\mathcal{P}_+^\uparrow$ [10]. We extend the previous analysis to **P**, **T** and **PT** as well here. The extension proves to be trivial in the absence of gauge fields. With gauge fields present, further analysis is needed especially as **T** is anti-unitary. We show in this paper that if for $\theta^{\mu\nu} = 0$, **P** and **T** are good symmetries in non-gauge theories, abelian gauge theories with or without matter fields, and non-abelian gauge theories without matter fields, then they continue to be so for $\theta^{\mu\nu} \neq 0$. But in non-abelian gauge theories with matter fields, such as the standard model, **P** and **CPT** are necessarily broken in scattering processes.

The behaviour of qft's on $\mathcal{A}_\theta(\mathbb{R}^N)$ under charge conjugation is not affected by $\theta^{\mu\nu}$. Thus **CPT** invariance is maintained in the qft's on $\mathcal{A}_\theta(\mathbb{R}^N)$ in non-gauge theories, abelian gauge theories with or without matter fields and non-abelian gauge theories without matter fields. That is so even though they violate many of the axioms of local quantum field theories.

Discrete transformations for qft's on noncommutative spacetimes have been analysed previously by Sheikh-Jabbari and by Álvarez-Gaumé and Vázquez-Mozo [11, 12]. The qft's they analysed are however formulated differently from the ones we study here.

2. A primer of past work

In this section, we summarize the pertinent aspects of our previous work for matter and gauge fields. While our treatment of matter fields is fully coherent with the work of Aschieri et al. [4], the two treatments differ in the treatment of gauge fields.

2.1 Matter fields without gauge interactions

We focus on a free complex scalar quantum field φ as an example. The discussion can be adapted to any free matter field.

The field φ on the GM plane $\mathcal{A}_\theta(\mathbb{R}^N)$ has the expansion

$$\varphi_\theta = \int d\mu(p) \, (a_{\mathbf{p}} \, e_p + b_{\mathbf{p}}^\dagger \, e_{-p}),$$

$$d\mu(p) = \frac{d^{N-1}p}{2p_0},$$

$$e_p(x) = e^{-ipx},$$

$$p_0 = \sqrt{\mathbf{p}^2 + m^2}, \quad m = \text{mass of } \varphi. \tag{2.1}$$

We set $N = 4$ for specificity. The operators $a_{\mathbf{p}}$, $b_{\mathbf{p}}$ can be written in terms of the annihiliation-creation operators $c_{\mathbf{p}}$, $d_{\mathbf{p}}$ for $\theta^{\mu\nu} = 0$ as follows using the "dressing transformation" [13, 14]:

$$a_{\mathbf{p}} = c_{\mathbf{p}} \, e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu}, \quad b_{\mathbf{p}} = d_{\mathbf{p}} \, e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu},$$

where

$$P_\mu = \int \frac{d^3p}{2p_0} \, (c_{\mathbf{p}}^\dagger c_{\mathbf{p}} + d_{\mathbf{p}}^\dagger d_{\mathbf{p}}) \, p_\mu = \text{Four-momentum operator.}$$

The commutation relations of $c_{\mathbf{p}}$, $c_{\mathbf{p}}^\dagger$, $d_{\mathbf{p}}$, $d_{\mathbf{p}}^\dagger$ are standard,

$$[c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger] = [d_{\mathbf{p}}, d_{\mathbf{q}}^\dagger] = 2p_0 \, \delta^3(\mathbf{p} - \mathbf{q}). \tag{2.2}$$

The remaining commutators involving these operators vanish.

The new operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are called the twisted or dressed operators and the map from c, d- to the a, b- operators is called dressing transformation. (The Grosse-Faddeev-Zamolodchikov algebra is a generalization of the above twisted or dressed algebra [13, 14]. See also [15] in this connection.)

Note that P_μ can also be written in terms of the twisted operators:

$$P_\mu = \int \frac{d^3p}{2p_0} \, (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) \, p_\mu = \text{Four-momentum.} \tag{2.3}$$

That is because $p_\mu \theta^{\mu\nu} P_\nu$ commutes with any of the operators for momentum p. For example $[P_\mu, a_{\mathbf{p}}] = -p_\mu a_{\mathbf{p}}$ so that $[p_\nu \theta^{\nu\mu} P_\mu, a_{\mathbf{p}}] = p_\nu \theta^{\nu\mu} p_\mu = 0$, θ being antisymmetric.

The antisymmetry of $\theta^{\mu\nu}$ allows us to write

$$c_{\mathbf{p}} e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu} = e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu} c_{\mathbf{p}}, \tag{2.4}$$

$$c_{\mathbf{p}}^\dagger e^{\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu} = e^{\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu} c_{\mathbf{p}}^\dagger. \tag{2.5}$$

Hence the ordering of factors here is immeterial.

It should also be noted that the map from c- to the a-operators is invertible,

$$c_{\mathbf{p}} = a_{\mathbf{p}} \, e^{\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu}, \quad d_{\mathbf{p}} = b_{\mathbf{p}} \, e^{\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu},$$

where P_μ is written as in eq. (2.3).

The $\star$-product between the fields is

$$(\varphi_\theta \star \varphi_\theta)(x) = \varphi_\theta(x) e^{\frac{i}{2}\overleftarrow{\partial} \wedge \overrightarrow{\partial}} \varphi_\theta(y)|_{x=y}, \tag{2.6}$$

$$\overleftarrow{\partial} \wedge \overrightarrow{\partial} := \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu.$$

The twisted quantum field φ differs from the untwisted quantum field φ_0 in two ways: *i.*) $e_p \in \mathcal{A}_\theta(\mathbb{R}^4)$ and *ii.*) $a_{\mathbf{p}}$ is twisted by statistics.

Both can be accounted by writing [17]

$$\varphi_\theta = \varphi_0 \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}, \tag{2.7}$$

where P_μ is the total momentum operator. From this follows that the $\star$-product of an arbitrary number of fields $\varphi_\theta^{(i)}$ ($i = 1, 2, 3, \cdots$) is

$$\varphi_\theta^{(1)} \star \varphi_\theta^{(2)} \star \cdots = (\varphi_0^{(1)}\varphi_0^{(2)}\cdots) \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}. \tag{2.8}$$

Although the rule (2.7) is for a spin-zero massive scalar field, we can apply it to all bosonic and fermionic matter fields with any spin. This is very convenient when we write the interaction Hamiltonian involving matter fields.

We can write the interaction Hamiltonian density for a pure matter field as

$$\mathcal{H}_{I\theta} = \mathcal{H}_{I0} \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \tag{2.9}$$

on using eq. (2.8). Thus statistics untwists the $\star$ in $\mathcal{H}_{I\theta}$. This is what leads to the θ-independence of S-operator in the absence of gauge fields [7–9, 17]. Interaction terms involving matter fields are always in this form.

2.2 Matter fields with gauge interactions

This section is based on [17].

We assume that the gauge (and gravity) fields are associated with the "commutative manifold" $\mathcal{A}_0(\mathbb{R}^4)$ whereas for Aschieri et al. [4] they are associated with $\mathcal{A}_\theta(\mathbb{R}^4)$. Matter fields on $\mathcal{A}_\theta(\mathbb{R}^4)$ must be transported by the connection compatibly with eq. (2.7), so a natural choice for the covariant derivative is [17]

$$D_\mu \varphi_\theta = (D_\mu^c \varphi_0) \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}, \tag{2.10}$$

where

$$D_\mu^c \varphi_0 = \partial_\mu \varphi_0 + A_\mu \varphi_0 \,, \tag{2.11}$$

P_μ is the total momentum operator for all the fields and

$$A_\mu \varphi_0(x) = A_\mu(x)\varphi_0(x) \ \text{[point-wise multiplication].} \tag{2.12}$$

This is indeed the correct choice of D_μ as it preserves the statistics, Poincaré and gauge invariance, and the requirement that D_μ is associated with $\mathcal{A}_0(\mathbb{R}^N)$:

$$[D_\mu, D_\nu]\varphi_\theta = \Big([D_\mu^c, D_\nu^c]\varphi_0\Big)e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \tag{2.13}$$

$$= \Big(F_{\mu\nu}^c \varphi_0\Big)e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}. \tag{2.14}$$

As $F_{\mu\nu}^c$ is the standard $\theta^{\mu\nu} = 0$ curvature, our gauge field is associated with $\mathcal{A}_0(\mathbb{R}^N)$. (For Aschieri et al. [4] the curvature would be the $\star$-commutator of D_μ's.) The gauge theory

formulation we adopt here is fully explained in [17]. It differs from the formulation of Aschieri et al. [4] (where covariant derivative is defined using star product) and has the advantage of being able to accomodate any gauge group and not just $U(N)$ gauge groups and their direct products. The gauge theory formulation we adopt here avoids multiplicity of fields that the expression for covariant derivates with $\star$ product entails.

In the single-particle sector (obtained by taking the matrix element of eq. (2.10) between vacuum and one-particle states), the P term can be dropped and we get for a single particle wave function f of a particle associated with φ_θ,

$$D_\mu f(x) = \partial_\mu f(x) + A_\mu(x) f(x). \tag{2.15}$$

Note that we can also write $D_\mu \varphi_\theta$ using $\star$-product:

$$D_\mu \varphi_\theta = \left(D^c_\mu e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \right) \star \left(\varphi_0 e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \right). \tag{2.16}$$

Our choice of covariant derivative allows us to write the interaction Hamiltonian density for pure gauge fields as follows:

$$\mathcal{H}^G_{I\theta} = \mathcal{H}^G_{I0}. \tag{2.17}$$

For a theory with matter and gauge fields, the interaction Hamiltonian density splits into

$$\mathcal{H}_{I\theta} = \mathcal{H}^{M,G}_{I\theta} + \mathcal{H}^G_{I\theta}, \tag{2.18}$$

where

$$\begin{aligned} \mathcal{H}^{M,G}_{I\theta} &= \mathcal{H}^{M,G}_{I0} \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}, \\ \mathcal{H}^G_{I\theta} &= \mathcal{H}^G_{I0}. \end{aligned} \tag{2.19}$$

The matter-gauge field couplings are also included in $\mathcal{H}^{M,G}_{I\theta}$.

In quantum electrodynamics (QED), $\mathcal{H}^G_{I\theta} = 0$. Thus the S-matrix for the twisted QED is the same for the untwisted QED:

$$S^{\text{QED}}_\theta = S^{\text{QED}}_0. \tag{2.20}$$

In a non-abelian gauge theory, $\mathcal{H}^G_\theta = \mathcal{H}^G_0 \neq 0$, so that in the presence of nonsinglet matter fields [17].

$$S^{M,G}_\theta \neq S^{M,G}_0. \tag{2.21}$$

3. On C, P, T and CPT

In this section we investigate **C**, **P**, **T** and **CPT** for $\mathcal{A}_\theta(\mathbb{R}^N)$. The **CPT** theorem [16] is very fundamental in nature and all local relativistic quantum field theories are **CPT** invariant. Qft's on the GM plane are non-local and so it is important to investigate the validity of the **CPT** theorem in these theories.

Here we first recall a fundamental result of earlier work [17] that **C** and the Poincaré group transform $c_{\mathbf{k}}$'s and $d_{\mathbf{k}}$'s and their adjoints as in the untwisted theories. The induced

transformations on the fields automatically imply the twisted coproduct in the matter sector, and of course the untwisted coproduct for gauge fields. This simple rule is proved for $\mathcal{P}_+^\uparrow$ in [17]. It then implies the same rule for the full group generated by $\mathbf{C}$ and $\mathcal{P}$ by the group properties of that group. (We always try to preserve such group properties.) This rule is repeatedly used below.

The matrix $\theta^{\mu\nu}$ is a constant antisymmetric matrix. We emphasise that in the approach using the twisted coproduct for the Poincaré group, $\theta^{\mu\nu}$ is *not* transformed by Poincaré transformarions or in fact by any other symmetry: they are truly constants. Nevertheless Poincaré invariance and other symmetries can be certainly recovered for Lagrangians invariant under the twisted symmetry actions at the level of classical actions and Wightman functions [1, 4, 5, 9].

3.1 Transformation of quantum fields under C, P and T

3.1.1 Charge conjugation C

The coproduct [3, 4] for the charge conjugation operator $\mathbf{C}$ in the twisted case is the same as the coproduct for $\mathbf{C}$ in the untwisted case since the charge conjugation operator commutes with P_μ. So, we write

$$\Delta_\theta(\mathbf{C}) = \Delta_0(\mathbf{C}) = \mathbf{C} \otimes \mathbf{C}. \tag{3.1}$$

Under charge conjugation,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{C}} d_{\mathbf{k}}, \ a_{\mathbf{k}} \xrightarrow{\mathbf{C}} b_{\mathbf{k}} \tag{3.2}$$

where $a_{\mathbf{k}} = c_{\mathbf{k}} \, e^{-\frac{i}{2}k\wedge P}$ and $b_{\mathbf{k}} = d_{\mathbf{k}} \, e^{-\frac{i}{2}k\wedge P}$.

Products of quantum fields on $\mathcal{A}_\theta(\mathbb{R}^N)$ transform in the same way as they would on $\mathcal{A}_0(\mathbb{R}^N)$ under the $\mathbf{C}$ operation. Thus we have

$$\varphi_\theta \xrightarrow{\mathbf{C}} \varphi_0^{\mathbf{C}} \, e^{\frac{1}{2}\overleftrightarrow{\partial}\wedge P}, \ \ \varphi_0^{\mathbf{C}} = \mathbf{C}\varphi_0\mathbf{C}^{-1}. \tag{3.3}$$

while the product of two such fields φ and χ transforms according to

$$\begin{aligned} \varphi_\theta \star \chi_\theta &= (\varphi_0\chi_0) \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \\ &\xrightarrow{\mathbf{C}} (\mathbf{C}\varphi_0\chi_0\mathbf{C}^{-1}) \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \\ &= (\varphi_0^{\mathbf{C}}\chi_0^{\mathbf{C}}) \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}. \end{aligned} \tag{3.4}$$

3.1.2 Parity P

Parity is a unitary operator on $\mathcal{A}_0(\mathbb{R}^N)$. But parity transformations do not induce automorphisms of $\mathcal{A}_\theta(\mathbb{R}^N)$ [19] if its coproduct is

$$\Delta_0(\mathbf{P}) = \mathbf{P} \otimes \mathbf{P}. \tag{3.5}$$

That is the coproduct is not compatible with the $\star$-product. Hence the coproduct for parity is not the same as that for the $\theta^{\mu\nu} = 0$ case.

But the twisted coproduct Δ_θ, where

$$\Delta_\theta(\mathbf{P}) = \mathcal{F}_\theta^{-1} \, \Delta_0(\mathbf{P}) \, \mathcal{F}_\theta, \tag{3.6}$$

is compatible with the $\star$-product. So, for $\mathbf{P}$ as well, compatibility with the $\star$-product fixes the coproduct [7].

Under parity,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{P}} c_{-\mathbf{k}}, \quad d_{\mathbf{k}} \xrightarrow{\mathbf{P}} d_{-\mathbf{k}} \tag{3.7}$$

and hence

$$a_{\mathbf{k}} \xrightarrow{\mathbf{P}} a_{-\mathbf{k}} \, e^{i(k_0\theta^{0i}P_i - k_i\theta^{i0}P_0)}, \quad b_{\mathbf{k}} \xrightarrow{\mathbf{P}} b_{-\mathbf{k}} \, e^{i(k_0\theta^{0i}P_i - k_i\theta^{i0}P_0)} \tag{3.8}$$

By an earlier remark [17], eqs. (3.7) and (3.8) imply the transformation law for twisted scalar fields. A twisted complex scalar field φ_θ transforms under parity as follows,

$$
\begin{aligned}
\varphi_\theta &= \varphi_0 \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \\
&\xrightarrow{\mathbf{P}} \mathbf{P}\Big(\varphi_0 \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}\Big)\mathbf{P}^{-1} \\
&= \varphi_0^{\mathbf{P}} \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge (P_0, -\overrightarrow{P})},
\end{aligned}
\tag{3.9}
$$

where $\varphi_0^{\mathbf{P}} = \mathbf{P}\varphi_0\mathbf{P}^{-1}$ and $\overleftarrow{\partial} \wedge (P_0, -\overrightarrow{P}) := -\overleftarrow{\partial}_0\theta^{0i}P_i - \overleftarrow{\partial}_i\theta^{ij}P_j + \overleftarrow{\partial}_i\theta^{i0}P_0$.

The product of two such fields φ_θ and χ_θ transforms according to

$$
\begin{aligned}
\varphi_\theta \star \chi_\theta &= (\varphi_0\chi_0) \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \\
&\xrightarrow{\mathbf{P}} (\varphi_0^{\mathbf{P}} \chi_0^{\mathbf{P}}) \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge (P_0, -\overrightarrow{P})}
\end{aligned}
\tag{3.10}
$$

Thus fields transform under $\mathbf{P}$ with an extra factor $e^{-(\overleftarrow{\partial}_0\theta^{0i}P_i + \partial_i\theta^{ij}P_j)} = e^{-\overleftarrow{\partial}_\mu\theta^{\mu j}P_j}$ when $\theta^{\mu\nu} \neq 0$.

3.1.3 Time reversal T

Time reversal $\mathbf{T}$ is an anti-linear operator. Due to antilinearity, $\mathbf{T}$ induces automorphisms on $\mathcal{A}_\theta(\mathbb{R}^N)$ for any $\theta^{\mu\nu}$ as well [19].

Under time reversal,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{T}} c_{-\mathbf{k}}, \qquad d_{\mathbf{k}} \xrightarrow{\mathbf{T}} d_{-\mathbf{k}} \tag{3.11}$$

$$a_{\mathbf{k}} \xrightarrow{\mathbf{T}} a_{-\mathbf{k}} \, e^{-i(k_i\theta^{ij}P_j)}, \quad b_{\mathbf{k}} \xrightarrow{\mathbf{T}} b_{-\mathbf{k}} \, e^{-i(k_i\theta^{ij}P_j)}. \tag{3.12}$$

Compatibility with the $\star$-product fixes the coproduct for $\mathbf{T}$ to be

$$\Delta_\theta(\mathbf{T}) = \mathcal{F}_\theta^{-1} \, \Delta_0(\mathbf{T}) \, \mathcal{F}_\theta. \tag{3.13}$$

This coproduct is also required in order to maintain the group properties of $\mathcal{P}$, the full Poincaré group.

A twisted complex scalar field φ_θ hence transforms under time reversal as follows,

$$
\begin{aligned}
\varphi_\theta &= \varphi_0 \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \\
&\xrightarrow{\mathbf{T}} \mathbf{T}\Big(\varphi_0 \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}\Big)\mathbf{T}^{-1} \\
&= \varphi_0^{\mathbf{T}} \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge (P_0, -\overrightarrow{P})},
\end{aligned}
\tag{3.14}
$$

while the product of two such fields φ_θ and χ_θ transforms according to

$$\begin{aligned}
\varphi_\theta \star \chi_\theta &= (\varphi_0 \chi_0)\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \\
&\xrightarrow{\ \mathbf{T}\ } (\varphi_0^{\mathbf{T}} \chi_0^{\mathbf{T}})\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge (P_0, -\overrightarrow{P})}
\end{aligned} \tag{3.15}$$

Thus the time reversal operation as well induces an extra factor $e^{-\overleftarrow{\partial}_i \theta^{ij} P_j}$ in the transformation property of fields when $\theta^{\mu\nu} \neq 0$.

3.1.4 CPT

When **CPT** is applied,

$$c_{\mathbf{k}} \xrightarrow{\ \mathbf{CPT}\ } d_{\mathbf{k}}, \qquad\qquad d_{\mathbf{k}} \xrightarrow{\ \mathbf{CPT}\ } c_{\mathbf{k}} \tag{3.16}$$

$$a_{\mathbf{k}} \xrightarrow{\ \mathbf{CPT}\ } b_{\mathbf{k}} e^{i(k \wedge P)}, \quad b_{\mathbf{k}} \xrightarrow{\ \mathbf{CPT}\ } a_{\mathbf{k}} e^{i(k \wedge P)}. \tag{3.17}$$

The coproduct for **CPT** is of course

$$\Delta_\theta(\mathbf{CPT}) = \mathcal{F}_\theta^{-1}\, \Delta_0(\mathbf{CPT})\, \mathcal{F}_\theta. \tag{3.18}$$

A twisted complex scalar field φ_θ transforms under **CPT** as follows,

$$\begin{aligned}
\varphi_\theta &= \varphi_0\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \\
&\xrightarrow{\ \mathbf{CPT}\ } \mathbf{CPT}\Big(\varphi_0\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}\Big)(\mathbf{CPT})^{-1} \\
&= \varphi_0^{\mathbf{CPT}}\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P},
\end{aligned} \tag{3.19}$$

while the product of two such fields φ_θ and χ_θ transforms according to

$$\begin{aligned}
\varphi_\theta \star \chi_\theta &= (\varphi_0 \chi_0)\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P} \\
&\xrightarrow{\ \mathbf{CPT}\ } (\varphi_0^{\mathbf{CPT}} \chi_0^{\mathbf{CPT}})\, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}.
\end{aligned} \tag{3.20}$$

4. CPT in non-abelian gauge theories

The standard model, a non-abelian gauge theory, is **CPT** invariant, but it is not invariant under **C**, **P**, **T** or products of any two of them. So we focus on discussing just **CPT** for its S-matrix when $\theta^{\mu\nu} \neq 0$. We also cover quantum electrodynamics (QED) by brief remarks. The discussion here can be easily adapted to any other non-abelian gauge theory.

4.1 Matter fields and their couplings to gauge fields

The interaction representation S-matrix is

$$S_\theta^{M,G} = \mathrm{T}\, exp\left[-i \int d^4x\, \mathcal{H}_{I\theta}^{M,G}(x)\right] \tag{4.1}$$

where $\mathcal{H}_{I\theta}^{M,G}$ is the interaction Hamiltonian density for matter fields (including also matter-gauge field couplings). Under **CPT**,

$$\mathcal{H}_{I\theta}^{M,G}(x) \xrightarrow{\ \mathbf{CPT}\ } \mathcal{H}_{I\theta}^{M,G}(-x) e^{\overleftarrow{\partial} \wedge P} \tag{4.2}$$

where $\overleftarrow{\partial}$ has components $\frac{\overleftarrow{\partial}}{\partial x_\mu}$. We write $\mathcal{H}_{I\theta}^{M,G}$ as

$$\mathcal{H}_{I\theta}^{M,G} = \mathcal{H}_{I0}^{M,G}\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}. \tag{4.3}$$

Thus we can write the interaction Hamiltonian density after **CPT** transformation in terms of the untwisted interaction Hamiltonian density:

$$
\begin{aligned}
\mathcal{H}_{I\theta}^{M,G}(x) \;&\xrightarrow{\textbf{CPT}}\; \mathcal{H}_{I\theta}^{M,G}(-x)\, e^{\overleftarrow{\partial}\wedge P} \\
&= \mathcal{H}_{I0}^{M,G}(-x)\, e^{-\frac{1}{2}\overleftarrow{\partial}\wedge P}\, e^{\overleftarrow{\partial}\wedge P} \\
&= \mathcal{H}_{I0}^{M,G}(-x)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}
\end{aligned} \tag{4.4}
$$

Hence under **CPT**,

$$
\begin{aligned}
S_\theta^{M,G} \;&=\; \mathrm{T}\, exp\left[-i\int d^4x\, \mathcal{H}_{I0}^{M,G}(x)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}\right] \\
&\to\; \mathrm{T}\, exp\left[i\int d^4x\, \mathcal{H}_{I0}^{M,G}(x)\, e^{-\frac{1}{2}\overleftarrow{\partial}\wedge P}\right] \\
&=\; (S_{-\theta}^{M,G})^{-1}
\end{aligned}
$$

But it has been shown elsewhere that $S_\theta^{M,G}$ is independent of θ [8]. Hence also $S_\theta^{M,G}$ is independent of θ.

Therefore a qft with no pure gauge interaction is **CPT** "invariant" on $\mathcal{A}_\theta(\mathbb{R}^N)$. In particular quantum electrodynamics (QED) preserves **CPT**.

4.2 Pure gauge fields

The interaction Hamiltonian density for pure gauge fields is independent of $\theta^{\mu\nu}$ in the approach of [17]:

$$\mathcal{H}_{I\theta}^G = \mathcal{H}_{I0}^G\,. \tag{4.5}$$

Hence also the S-matrix becomes θ-independent,

$$S_\theta^G = S_0^G\,, \tag{4.6}$$

and **CPT** holds as a good "symmetry" of the theory.

4.3 Matter and gauge fields

All interactions of matter and gauge fields can be fully discussed by writing the S-matrix as

$$\mathbf{S}_\theta^{M,G} = \mathrm{T}\, exp\left[-i\int d^4x\, \mathcal{H}_{I\theta}(x)\right], \tag{4.7}$$

$$\mathcal{H}_{I\theta} = \mathcal{H}_{I\theta}^{M,G} + \mathcal{H}_{I\theta}^G, \tag{4.8}$$

where

$$\mathcal{H}_{I\theta}^{M,G} = \mathcal{H}_{I0}^{M,G}\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$
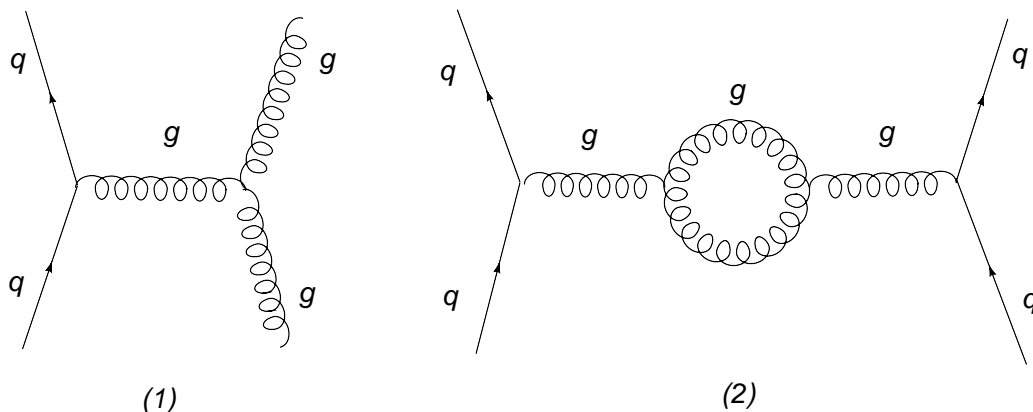
Figure 1: CPT violating processess on GM plane. (1) shows quark-gluon scattering with a three-gluon vertex. (2) shows a gluon-loop contribution to quark-quark scattering. (Calculation of such processes are being attempted).

and

$$\mathcal{H}_{I\theta}^{G} = \mathcal{H}_{I0}^{G}.$$

In QED, $\mathcal{H}_{I\theta}^{G} = 0$. Thus the S-matrix $\mathbf{S}_{\theta}^{\text{QED}}$ is the same as for the $\theta^{\mu\nu} = 0$ case in QED:

$$\mathbf{S}_{\theta}^{\text{QED}} = \mathbf{S}_{0}^{\text{QED}}. \tag{4.9}$$

Hence **C**, **P**, **T** and **CPT** are good "symmetries" for QED on the GM plane.

For a non-abelian gauge theory with non-singlet matter fields, $\mathcal{H}_{I\theta}^{G} = \mathcal{H}_{I0}^{G} \neq 0$ so that if $\mathbf{S}_{\theta}^{M,G}$ is the S-matrix of the theory,

$$\mathbf{S}_{\theta}^{M,G} \neq \mathbf{S}_{0}^{M,G}. \tag{4.10}$$

The S-matrix $\mathbf{S}_{\theta}^{M,G}$ depends only on θ^{0i} in a non-abelian theory (see section V), that is, $\mathbf{S}_{\theta^{\mu\nu}}^{M,G} = \mathbf{S}_{\theta^{0i}}^{M,G}$. Applying **C**, **P** and **T** on $\mathbf{S}_{\theta}^{M,G}$ we can see that **C** and **T** do not affect θ^{0i} while **P** changes its sign. Thus a non-zero θ^{0i} contributes to **P** and **CPT** violation.

4.4 On QED

The S-matrix of QED is invariant under **C**, **P** and **T** for $\theta^{\mu\nu} = 0$. But as remarked earlier, its S-matrix is independent of $\theta^{\mu\nu}$. Hence QED is also **C**, **P** and **T** invariant for any $\theta^{\mu\nu}$.

5. On Feynman graphs

In non-abelian gauge theories, $\mathcal{H}_{I\theta}^{G} = \mathcal{H}_{I0}^{G}$ is not zero as gauge fields have self-interactions. The preceding discussions show that the effects of $\theta^{\mu\nu}$ can show up only in Feynman diagrams which are sensitive to products of $\mathcal{H}_{I\theta}^{M,G}$'s with $\mathcal{H}_{I0}^{G}$'s. Figure 1 shows two such diagrams.

As an example, consider the first diagram in figure 1. To lowest order, it depends on θ^{0i}.

We can substitute eq. (4.3) for $\mathcal{H}_{I\theta}^{M,G}$ and integrate over $\mathbf{x}$. That gives,

$$\mathbf{S}^{(2)} = -\frac{1}{2}\int d^4x\, d^4y\; \mathrm{T}\Big(H_{I0}^{M,G}(x)\, e^{\frac{1}{2}\overleftarrow{\partial}_0\theta^{0i}P_i} H_{I0}^{G}(y)\Big)$$

where $\overleftarrow{\partial}_0$ acts *only* on $H_{I0}^{M,G}(x)$ (and not on the step functions in time entering in the definition of T.)

Now P_i, being component of spatial momentum, commutes with

$$\int d^3y\; \mathcal{H}_{I0}^{G}(y)$$

and hence for computing the matrix element defining the process (*1*) in figure 1, we can substitute $\overrightarrow{P}_{\mathrm{in}}$ for $\overrightarrow{P}$, $\overrightarrow{P}_{\mathrm{in}}$ being the total incident spatial momentum:

$$\mathbf{S}^{(2)} = -\frac{1}{2}\int d^4x\, d^4y\; \mathrm{T}\Big(H_{I0}^{M,G}(x)\, e^{\frac{1}{2}\overleftarrow{\partial}_0\theta^{0i}P_i^{\mathrm{in}}} H_{I0}^{G}(y)\Big). \tag{5.1}$$

Thus $\mathbf{S}^{(2)}$ depends on θ^{0i} unless

$$\theta^{0i}P_i^{\mathrm{in}} = 0. \tag{5.2}$$

This will happen in the center-of-mass system or more generally if $\overrightarrow{\theta^0}=(\theta^{01},\,\theta^{02},\,\theta^{03})$ is perpendicular to $\overrightarrow{P}^{\mathrm{in}}$.

Under $\mathbf{P}$ and $\mathbf{CPT}$, $\theta^{0i}\to -\theta^{0i}$. This shows clearly that in a general frame, θ^{0i} contributes to $\mathbf{P}$ violation and causes $\mathbf{CPT}$ violation.

The dependence of $S^{(2)}$ on the incident total spatial momentum shows that the scattering matrix is not Lorentz invariant. This noninvariance is caused by the nonlocality of the interaction Hamiltonian density: if we evaluate it at two spacelike separated points, the resultant operators do not commute. Such a violation of causality can lead to Lorentz-noninvariant S-operators [17].

The reasoning which reduced $e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$ to $e^{\frac{1}{2}\overleftarrow{\partial}_0\theta^{0i}P_i^{\mathrm{in}}}$ is valid to all such factors in an arbitrary order in the perturbation expansion of the S-matrix and for arbitrary processes, $\overrightarrow{P}^{\mathrm{in}}$ being the total incident spatial momentum. As $\theta^{\mu\nu}$ occur only in such factors, this leads to an interesting conclusion: if the scattering happens in the center-of-mass frame, or any frame where $\theta^{0i}P_i^{\mathrm{in}} = 0$, then the θ-dependence goes away from the S-matrix. That is, P and CPT remain intact if $\theta^{0i}P_i^{\mathrm{in}} = 0$. The theory becomes P and CPT violating in all other frames.

6. Experimental signals

Terms with products of $\mathcal{H}_{I\theta}^{M,G}$ and $\mathcal{H}_{I\theta}^{G}$ are θ-dependent and they violate $\mathbf{CPT}$. Electroweak and QCD processes will acquire dependence on θ. This is the case when a diagram involves products of $\mathcal{H}_{I\theta}^{M,G}$ and $\mathcal{H}_{I\theta}^{G}$. For example quark-gluon and quark-quark scattering on the GM plane become θ-dependent $\mathbf{CPT}$ violating processes (See figure 1). This may be tested experimentally. Possibilities in this direction are being explored.

7. Conclusions

We have examined the discrete symmetries **C**, **P**, **T** and **CPT** for qft's on the GM plane; showing how they are modified by the twisted statistics of quantum fields. Twisted statistics is required by Lorentz invariance. We have shown that the action of these discrete symmetries on the S-matrix is independent of $\theta^{\mu\nu}$ in particular when matter and non-abelian gauge fields interacting with each other are not present. However in the presence of such matter and gauge fields, the S-matrix violates **P** and **CPT**. (It violates also Lorentz invariance [17].) We have also mentioned some processes in which the θ-dependence is apparent. If the scattering happens in the center-of-mass frame, or any frame where $\theta^{0i}P_i^{\text{in}} = 0$, then the θ-dependance goes away from the S-matrix. **P** and **CPT** remain intact if $\theta^{0i}P_i^{\text{in}} = 0$.

Acknowledgments

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